A cohomological construction of integrable hierarchies of hydrodynamic type

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Abstract

We explain how to use the theory of bidifferential ideals to construct integrable hierarchies of hydrodynamic type

1 Introduction

In this note we show how to use the theory of bidifferential ideals to solve a classical problem of the theory of first-order partial differential equations of hydrodynamic type

$$\frac{\partial u}{\partial t} = A(u)\frac{\partial u}{\partial x}$$

in two independent variables (x, t) and n dependent variables $u = (u^1, ..., u^n)$ with periodic boundary conditions. The problem is to construct an infinite family

$$\frac{\partial u}{\partial t_i} = A_i(u)\frac{\partial u}{\partial x}$$

of equations of this type whose flows commute in pair, so that each equation can be considered as defining a symmetry of all the remaining equations. Simultaneously the problem requires to construct an infinite family of functionals

$$I_l[u] = \int_{S^1} k_l(u)dx$$

which are constants of motion of all the previous flows, so that

$$\frac{\partial I_l}{\partial t_k} = 0$$

for all values of the indexes $l, k \in \mathbb{N}$.

As usual we consider a system of hydrodynamic type as a vector field on a loop space $\hat{\mathcal{F}}$, whose points are the C^∞-maps from the circle to a base manifold $\mathcal{F}$ endowed with coordinates $(u^1, ..., u^n)$. On this manifold the problem requires to construct an infinite family of pairwise commuting vector fields. A simple way for dealing with this problem is to consider a linear operator $N : T\hat{\mathcal{F}} \to T\hat{\mathcal{F}}$, called a recursion operator, and to define the family of commuting vector fields X_j through the recursive relation

$$X_{j+1}(u) = N_u X_j(u).$$

It is known that, if the torsion of N vanishes, and if the first vector field X_0 of the hierarchy is chosen in such a way that the Lie derivative of N along X_0 vanishes, then all the vector fields X_j commute in pair.

Although conceptualy simple, the previous scheme has several practical drawbacks, the main of which is that the recursion operator N is often a nonlocal operator (that is an integro-differential operator), so that it is extremely difficult to give the theory a sound formal basis. Moreover, there is no reason to restrict our attention to such simple recursion relations as that written before. That would entail a noticeable loss of interesting examples.

In this note we show how to amend the theory of recursion operator of both drawbacks (at least in the setting of PDEs of hydrodynamic type). Our theory rests on two simple ideas. The first is that the recursion operator $N : T\hat{\mathcal{F}} \to T\hat{\mathcal{F}}$, on the loop space, may be conveniently replaced by a much simpler recursion operator $L : T\mathcal{F} \to T\mathcal{F}$ on the base manifold $\mathcal{F}$. The second idea is that the simple recursion relation of Lenard's type is just a particular instance of a general class of recursion relations provided by the theory of "prolongation of bidifferential ideals". Despite the exotic language, the new scheme of iteration is very simple, and it may be of use for more general class of equations than those of hydrodynamic type.

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2 Bidifferential ideals and hydrodynamic type hierarchies

A tensor field $L : T\mathcal{F} \to T\mathcal{F}$, of type $(1,1)$ on a manifold $\mathcal{F}$, of dimension n, is torsionless if the following identity

$$[LX, LY] - L[LX, Y] - L[X, LY] + L^2[X, Y] = 0$$

is verified for any pair of vector fields X and Y on $\mathcal{F}$, the bracket denoting the standard commutator of vector fields. One of the reasons to consider a torsionless tensor field

L is that it defines, according to the theory of graded derivations of Frölicher-Nijenhuis (see [1]), a differential operator d_L, of degree 1 and type d, on the Grasmann algebra of differential forms on $\mathcal{F}$, verifying the fundamental conditions

$$d \cdot d_L + d_L \cdot d = 0 \qquad\qquad d_L^2 = 0.$$

On functions and 1-forms this derivation is defined by the following equations

$$\begin{aligned} &d_L f(X) = df(LX) \\ &d_L \alpha(X,Y) = L_{LX}(\alpha(Y)) - L_{LY}(\alpha(X)) - \alpha([X,Y]_L), \end{aligned}$$

where

$$[X,Y]_L = [LX,Y] + [X,LY] - L[X,Y].$$

Since the pair of differential operators d and d_L define a double cohomological complex, one may introduce the concept of bidifferential ideal of forms.

Definition 1 *A bidifferential ideal $\mathfrak{I}$ is an ideal of differential forms on $\mathcal{F}$ which is closed with respect to the action of both d and d_L:*

$$d(\mathfrak{I}) \subset \mathfrak{I} \qquad\qquad d_L(\mathfrak{I}) \subset \mathfrak{I}$$

Let us make this concept concrete by a simple example. Assume that $\mathfrak{I}$ has rank 1, and therefore that is generated by a single 1-form α. The above equations then require that there exist two 1-forms λ and μ (called the Frobenius multipliers) such that

$$d\alpha = \lambda \wedge \alpha \qquad\qquad d_L \alpha = \mu \wedge \alpha.$$

In virtue of the first condition and on account of the Frobenius theorem, one may assume, without loss of generality, that the 1-form α is exact: $\alpha = dh$. Therefore any bidifferential ideal of rank 1 is defined by a function h for which there exists a 1-form μ such that

$$dd_L h = \mu \wedge dh.$$

Due to the basic identities $d^2 = 0$ and $d_L^2 = 0$, the 1-form μ must verify the pair of conditions

$$d\mu \wedge dh = 0 \qquad\qquad d_L \mu \wedge dh = 0.$$

A special class of ideals is selected by imposing that the multiplier μ obeys the stronger conditions

$$d\mu = 0 \qquad\qquad d_L \mu = 0.$$

Even more stringently, one may assume that μ is an exact one form, $\mu = da$, and that its potential a verifies the remaining condition

$$dd_L a = 0. \tag{1}$$

By this process of successive restrictions, one thus selects a special class of bidifferential ideals that are called flat.

Definition 2 *A flat bidifferential ideal $\mathfrak{I}$, of rank 1, on a manifold $\mathcal{F}$ endowed with a torsionless tensor field $L : T\mathcal{F} \to T\mathcal{F}$, is the ideal of forms generated by the differential dh of a function $h : \mathcal{F} \to \mathcal{F}$ obeying the condition*

$$dd_L h = da \wedge dh \tag{2}$$

with respect to a conformal factor a which in turn verifies the cohomological condition (1).

Since any solution of (1) is also a solution of the associated equation (2) there exists a special class of bidifferential ideals generated by a single function a.

The main purpose of the present note is to prove that an integrable hierarchy of hydrodynamic type on $\hat{\mathcal{F}}$ is associated with any flat bidifferential ideal, of rank 1, on $\mathcal{F}$. Probably this correspondence may be extended to higher rank flat bidifferential ideals on $\mathcal{F}$ (to be defined in a similar way), but in this note we shall stick, for simplicity, to the rank 1 case. The key to understand the relation between bidifferential ideals on $\mathcal{F}$ and integrable hierarchies of hydrodynamic type on $\hat{\mathcal{F}}$ is the process of prolongation of bidifferential ideals which is presently introduced.

Let us consider again an arbitrary bidifferential ideals $\mathfrak{I}$ on $\mathcal{F}$ which is generated by the 1-forms $(\alpha_1, ..., \alpha_p)$, and let us denote by $\mathfrak{I}'$ the ideal which is generated by the same forms and by their iterations $(L^*\alpha_1, ..., L^*\alpha_p)$. One can prove that $\mathfrak{I}'$ is still a bidifferential ideal. So the family of all bidifferential ideals on $\mathcal{F}$ is closed under the action of L. Since $\mathfrak{I}'$ clearly includes the original ideal $\mathfrak{I}$, it is natural to refer to this process as the process of prolongation of bidifferential ideals. In particular, if $\mathfrak{I}$ is flat, one can show that also $\mathfrak{I}'$ is flat. This means that $\mathfrak{I}'$ is generated by a family of exact 1-forms dh_j, $j = 1, ..., q$ obeying the conditions

$$dd_L h_i = \sum_{k=1}^{q} da_j^k \wedge dh_k$$

with conformal factors a_j^k verifying the "flatness conditions"

$$dd_L a_j^k = \sum_{l=1}^{q} da_j^l \wedge da_l^k.$$

The proof of these claims in full generality is not difficult but is irrelevant for the purpose of the present note. So we limit ourselves to detail the process of prolongation of bidiferrential ideals in the particular case of rank 1 ideals ([2]).

Proposition 1 *Let $\mathfrak{I}$ be a rank 1 flat bidifferential ideal on $\mathcal{F}$, generated by a function h satisfying the condition (2) with a conformal factor a satisfying the flatness condition (1). Then the successive prolongations $\mathfrak{I}'$, $\mathfrak{I}''$,... of $\mathfrak{I}$ are flat bidifferential ideals of rank 2,3,... which are generated by the possibly infinite sequence of functions $(h_0, h_1, h_2, ...)$ recursively defined by the relations*

$$dh_{k+1} = d_L h_k - a_k dh_0$$
$$da_{k+1} = d_L a_k - a_k da_0$$

where $a_0 = a$ and $h_0 = h$.

Proof

Start from equations (2) and (1) written in the form

$$\begin{aligned} d(d_L h_0 - a_0 dh_0) &= 0 \\ d(d_L a_0 - a_0 da_0) &= 0. \end{aligned}$$

Locally there exists a pair of functions (h_1, a_1) such that

$$\begin{aligned} dh_1 &= d_L h_0 - a_0 dh_0 \\ da_1 &= d_L a_0 - a_0 da_0. \end{aligned}$$

Let us apply the differential d_L to both sides of these equations. One obtains

$$\begin{aligned} dd_L h_1 &= d_L a_0 \wedge dh_0 - a_0 dd_L h_0 \\ &= d_L a_0 \wedge dh_0 - a_0 da_0 \wedge dh_0 \\ &= da_1 \wedge dh_0 \end{aligned}$$

and

$$dd_L a_1 = d_L a_0 \wedge da_0 = da_1 \wedge da_0.$$

Let us now proceeds by induction, assuming that there exists a couple of functions (h_k, a_k) such that

$$\begin{aligned} dd_L h_k - da_k \wedge dh_0 &= 0 \\ dd_L a_k - da_k \wedge da_0 &= 0. \end{aligned}$$

These conditions imply the existence of a new pair of functions (h_{k+1}, a_{k+1}) such that

$$\begin{aligned} dh_{k+1} &= d_L h_k - a_k dh_0 \\ da_{k+1} &= d_L a_k - a_k da_0. \end{aligned}$$

By applying d_L to both sides, one readily obtains

$$\begin{aligned} dd_L h_{k+1} &= da_{k+1} \wedge dh_0 \\ dd_L a_{k+1} &= da_{k+1} \wedge da_0, \end{aligned}$$

so the iterative process starts again. The form of the iteration immediately shows that the differentials dh_k generate the successive prolongations of the original flat bidifferential ideal $\mathfrak{I}$ generated by the pair of functions (h_0, a_0). The proof that the prolonged ideals are still flat is omitted, since it is of no interest for the present note.

$\square$

The previous result clearly points out the relation between bidifferential ideals and chains. The functions defining a flat bidifferential ideal on $\mathcal{F}$ may be used as starting elements of appropriate recurrence relations defining, at each step, a new set of functions. These recurrence relations translate on functions the process of prolongation of bidifferential ideals. The results of each step of the iterative procedure define the members of

the chain associated with the original bidifferential ideal. The missing part of the present theory is to show that the chain of bidifferential ideal on $\mathcal{F}$, just described, may be converted into a chain of commuting vector fields of hydrodynamic type on the loop space $\hat{\mathcal{F}}$ related to $\mathcal{F}$.

We have all the tools required to perform this last step except two. The first is the class of tensor fields M_k, of type $(1,1)$, recursively defined by

$$M_{k+1} = M_k L - a_k E,$$

starting from the identity E. The first elements of this family are

$$\begin{aligned} M_0 &= E \\ M_1 &= L - a_0 E \\ M_2 &= L^2 - a_0 L - a_1 E \end{aligned}$$

and so on. They naturally arise in the present picture, since they are the linear operators relating the differentials (da_k, dh_k) of the iterated functions to the initial differentials (da_0, dh_0) through the relations

$$\begin{aligned} da_k &= M_k^* da_0 \\ dh_k &= M_k^* dh_0. \end{aligned}$$

The second missing element is a new chain of functions $(k_0, k_1, k_2, ...)$ which may be built from the functions h_k and a_k coming from the process of prolongation of bidifferential ideals. One may define the new functions either by specifying how they are related to (h_k, a_k) by means of relations of the form

$$\begin{aligned} k_0 &= h_0 \\ k_1 &= h_1 + a_0 h_0 \\ k_2 &= h_2 + a_0 h_1 + (a_1 + a_0^2) h_0 \\ k_3 &= h_3 + a_0 h_2 + (a_1 + a_0^2) h_1 + (a_2 + 2a_1 a_0 + a_0^3) \\ &\dots\dots\dots\dots \end{aligned}$$

or, more conveniently, by the recurrence relation

$$dk_{l+1} = d_L k_l + k_l da_0 \tag{3}$$

starting from $k_0 = h_0$. One may easily check that this recurrence relation provides, at each step, an exact 1-form, thus allowing to compute a new iterated function. Moreover, by applying d_L to both sides of (3) we obtain the condition

$$dd_L k_{l+1} = d_L k_l \wedge da_0 = dk_{l+1} \wedge da_0$$

which means that the differentials dk_0, dk_1, dk_2,... define a chain of flat bidifferential ideals having the same conformal factor a_0.

We can now state the main result of the note.

Proposition 2 *The chain of vector fields of hydrodynamic type defined on $\hat{\mathcal{F}}$ by the equations*

$$\frac{\partial u}{\partial t_j} = M_j u_x$$

commute in pair, and the chain of functionals

$$I_l = \int_{s^1} k_l(u) dx$$

are the related integrals. So any flat bidifferential ideal of rank 1 on $\mathcal{F}$ allows to construct an integrable hierarchy of hydrodynamic type on the loop space $\hat{\mathcal{F}}$ associated with $\mathcal{F}$.

Proof

In order to prove the first claim we recall (see [4]) that two vector fields $u_t = Au_x$ and $u_\tau = Bu_x$ commute if and only if the tensor fields of type $(1,1)$ A and B commute as linear operator

$$A \cdot B - B \cdot A = 0,$$

and verify the differential condition

$$[AX, BX] - A[X, BX] - B[AX, X] = 0 \tag{4}$$

for any vector field X on $\mathcal{F}$. In our case the first condition is obviously satisfied. It remains to prove the condition (4). To this end, it is suitable to introduce the symbol

$$\{AX, BX\} = [AX, BX] - A[X, BX] - B[AX, X] \tag{5}$$

to condense in the formula

$$\{M_{k+1}X, M_{l+1}X\} = L \cdot \{M_k X, M_{l+1}X\} + L \cdot \{M_{k+1}X, M_l X\} - L^2 \cdot \{M_k X, M_l X\} \tag{6}$$

a useful identity relating the four tensor field $(M_k, M_{k+1}, M_l, M_{l+1})$. This identity is proved in Appendix , by exploiting the vanishing of the torsion of L and the recurrence relations on the conformal factors a_k. The above identity holds for all integers $k, l \in \mathbb{N}$. Notice that for $M_0 = E$ one has

$$\{M_0 X, M_l X\} = 0,$$

for all $l \in \mathbb{N}$, meaning that the first vector field $\hat{X}_0 = M_0 u_x$ commute with all the remaining vector fields $\hat{X}_l = M_l u_x$ of the hierarchy:

$$[\hat{X}_0, \hat{X}_l] = 0.$$

Assume to have proved that, for a certain k, the vector field $\hat{X}_k = M_k u_x$ commute with all the vector fields of the hierarchy:

$$[\hat{X}_k, \hat{X}_l] = 0 \qquad \forall l \in \mathbb{N}.$$

This means to assume that

$$\{M_k X, M_l X\} = 0 \qquad \forall l \in \mathbb{N}.$$

Then the identity (6) shows that

$$\{M_{k+1} X, M_l X\} = L\{M_{k+1} X, M_{l-1} X\}$$

and, by iteration, that

$$\{M_{k+1} X, M_l X\} = L^l\{M_{k+1} X, M_0 X\} = 0$$

Therefore, if $\hat{X}_k$ commute with the hierarchy, also $\hat{X}_{k+1}$ commute with the hierarchy. This remark ends the proof of the first part of the theorem.

In order to prove the second part, it is sufficient to notice the following recursive relations on the derivatives of the functionals I_j:

$$\begin{aligned}
\hat{X}_{k+1}(I_j) &= \int_{S^1} < dk_j, L\hat{X}_k - a_k\hat{X}_0 > dx = \\
&= \int_{S^1} (< d_L k_j, \hat{X}_k > - < dk_j, a_k\hat{X}_0 > dx = \\
&= \int_{S^1} (< dk_{j+1} - k_j da_0, \hat{X}_k > - < dk_j, a_k\hat{X}_0 >)dx = \\
&= \int_{S^1} < dk_{j+1}, \hat{X}_k > dx - \int_{S^1} (< k_j da_k + a_k dk_j, \hat{X}_0 >= \\
&= \hat{X}_k(I_{j+1}) - \int_{S^1} < d(k_j a_k), \hat{X}_0 > dx = \\
&= \hat{X}_k(I_{j+1}) - \int_{S^1} \frac{d}{dx}(k_j a_k) dx = \\
&= \hat{X}_k(I_{j+1})
\end{aligned}$$

Therefore

$$\hat{X}_K(I_j) = \hat{X}_{k+1}(I_{j+1}) = ... = \hat{X}_0(I_{j+k}) = 0$$

since $\hat{X}_0(I_l) = 0$ for any l.

$\square$

3 The diagonal case

Let us restrict our attention to the case where $L : T\mathcal{F} \rightarrow T\mathcal{F}$ has real and distinct eigenvalues. On account of the vanishing of the torsion of L, this assumption entails the existence of a system of local coordinates $(q^1, ..., q^n)$ such that

$$L^* dq^i = f^i(q^i) dq^i$$

The basic condition (1), starting the iterative scheme of bidifferential ideals, takes the simpler form

$$\partial_i \partial_j a_0 = 0,$$

showing that the most general admissible conformal factor a_0 is a sum of functions of a single coordinate:

$$a_0 = \sum_i g^i(q^i).$$

It follows that the first equation of the hierarchy of equations of hydrodynamic type generated by a_0 has the form

$$q^i_t = (f^i(q) - a_0)q^i_x. \tag{7}$$

These equations have been previously considered by Pavlov, who obtained them through the study, in his language, of the finite-component reductions of the infinite momentum chain

$$\partial_t c_k = \partial_x c_{k+1} - c_1 \partial_x c_k \qquad k = 0, \pm 1, \pm 2, \ldots \qquad .$$

One can thus say that the construction of the previous section explains, in the case of a diagonalizable recursion operator L, the geometric meaning of the Pavlov's reductions.

It is known that the system (7) is semi-Hamiltonian, that is the characteristic speeds $v^i(q) = f^i - a_0$ verify the "complete integrability" conditions

$$\partial_j \left(\frac{\partial_k v^i}{v^k - v^i} \right) = \partial_k \left(\frac{\partial_j v^i}{v^j - v^i} \right) \qquad \forall i \neq j \neq k \neq i. \tag{8}$$

It is also known that the functional $I[u] = \int h(u)dx$ is a first integral of a semi-Hamiltonian system $q^i_t = v^i(q)q^i_x$ if and only if the density h satisfies the following system (see [5])

$$\partial_i \partial_j h - \frac{\partial_j v^i}{v^j - v^i}\partial_i h - \frac{\partial_i v^j}{v^i - v^j}\partial_j h = 0 \qquad i \neq j. \tag{9}$$

In order to link the present approach to the existing literature, we observe that in the present case, where $v^i = f^i - a_0$, the equations (8) and (9) read:

$$\begin{aligned} &\partial_i \partial_j a_0 = 0 \\ &(f^i - f^j)\frac{\partial^2 h}{\partial q^i \partial q^j} = \frac{\partial a_0}{\partial q^i}\frac{\partial h}{\partial q^j} - \frac{\partial a_0}{\partial q^j}\frac{\partial h}{\partial q^i}. \end{aligned}$$

They are nothing else than the coordinate form of the basic equations (1) and (2):

$$\begin{aligned} &dd_L a_0 = 0 \\ &dd_L h = da_0 \wedge dh \end{aligned}$$

This remark provides a new insight on the meaning of these equations. The thorough study of the non diagonal case is outsides the limits of the present note.

4 Appendix

This section is devoted to the proof of the identitity

$$\{M_{k+1}X, M_{l+1}X\} = L\cdot\{M_kX, M_{l+1}X\} + L\cdot\{M_{k+1}X, M_lX\} - L^2\cdot\{M_kX, M_lX\}$$

By using the relation $M_{k+1} = M_kL - a_kE$, we obtain

$$\begin{aligned}
&\{M_{k+1}X, M_{l+1}X\} = \\
&([LM_kX, LM_lX] - [a_kX, LM_lX] - [LM_kX, a_lX] + [a_kX, a_lX]) + \\
&+(-LM_k[X, M_{l+1}X] + a_k[X, M_{l+1}X]) + (-LM_l[M_{k+1}X, X] + a_l[M_{k+1}X, X])
\end{aligned}$$

Let us compute the sum of the first terms in each bracket:

$$\begin{aligned}
&[LM_kX, LM_lX] - LM_k[X, M_{l+1}X] - LM_l[M_{k+1}X, X] = \\
&[LM_kX, LM_lX] + L\cdot\{M_kX, M_{l+1}X\} - L[M_kX, M_{l+1}X] + LM_{l+1}[M_kX, X] + \\
&+L\cdot\{M_{k+1}X, M_lX\} - L[M_{k+1}X, M_lX] + LM_{k+1}[X, M_lX] = \\
&L\cdot(\{M_kX, M_{l+1}X\} + \{M_{k+1}X, M_lX\}) - L^2\cdot\{M_kX, M_lX\} - a_lL[M_kX, X] + \\
&-a_kL[X, M_lX] + L^2[M_kX, M_lX] + [LM_kX, LM_lX] - L[M_kX, LM_lX] + \\
&+L[M_kX, a_lX] - L[LM_kX, M_lX] + L[a_kX, M_lX]
\end{aligned}$$

Taking into account that (for the vanishing of the torsion of L)

$$[LM_kX, LM_lX] + L^2[M_kX, M_lX] - L[M_kX, LM_lX] - L[LM_kX, M_lX] = 0,$$

and that

$$\begin{aligned}
-a_lL[M_kX, X] + L[M_kX, a_lX] &=< M_kX, da_l > LX =< M_lM_kX, da_0 > LX \\
-a_kL[X, M_lX] + L[a_kX, M_lX] &=< M_lX, da_k > LX =< M_kM_lX, da_0 > LX
\end{aligned}$$

we obtain finally

$$\begin{aligned}
&[LM_kX, LM_lX] - LM_k[X, M_{l+1}X] - LM_l[M_{k+1}X, X] = \\
&L\cdot\{M_kX, M_{l+1}X\} + L\cdot\{M_{k+1}X, M_lX\} - L^2\cdot\{M_kX, M_lX\}
\end{aligned}$$

It remains to prove that the sum of the remaining terms vanishes. But this follows from the following sequence of identities:

$$\begin{aligned}
&-[a_kX, LM_lX] - [LM_kX, a_lX] + [a_kX, a_lX] + a_k[X, M_{l+1}X] + a_l[M_kX, X] = \\
&-[a_kX, M_{l+1}X] + a_k[X, M_{l+1}X] - [M_{k+1}X, a_lX] + a_l[M_{k+1}X, X] - [a_kX, a_lX] = \\
&< M_{l+1}X, da_k > X - < M_{k+1}X, da_l > X + a_l < X, da_k > X - a_k < X, da_l > X = 0
\end{aligned}$$

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